Exam. Code : 211001
Subject Code : 3836

M.Sc. Mathematics Ist Semester

MATH-553 ALGEBRA–I

Time Allowed—3 Hours] [Maximum Marks—100

Note :— Candidates are required to attempt **five** questions, selecting at least **one** question from each Section. The **fifth** question may be attempted from any Section.

SECTION—A

1. (a) Let H and K be two subgroups of a group G. Then HK is subgroup of G if and only if HK=KH. 6

   (b) The intersection of two subgroups of finite index is of finite index. 4

   (c) State and prove Lagrange's theorem and prove that for every $a \in G$, $o(a) \mid n$, where n is order of G. 10

2. (a) Every cyclic group is isomorphic to $\mathbb{Z}$ or to $\frac{z}{<n>}$ for some $n \in \mathbb{N}$. 7

   (b) Give an example of a group G having subgroups K and T such that K is normal in T and T is normal in G but K is not a normal subgroup of G. 3

(c) Prove that a non-abelian group of order 6 is isomorphic to S_3. 10

SECTION—B

3. (a) Show that each dihedral group is homomorphic to the group of order 2. 5

(b) Find Aut(K) where K is the Klein four-group. 5

(c) If a permutation $\sigma \in S_n$ is a product of r transpositions and also a product of s transpositions, then r and s are either both even or both odd. 10

4. (a) Show that A_n is simple for all $n \geq 5$. 10

(b) Show that the group $\mathbb{Z}_8$ cannot be written as the direct sum of two nontrivial subgroups. 5

(c) Prove that there is a 1-1 correspondence between the family F of nonisomorphic abelian groups of order p^e, p prime and the set P(e) of partitions of e. 5

SECTION—C

5. (a) Let G be a group containing an element of finite order $n > 1$ and exactly two conjugacy classes. Prove that $| G | = 2$. 7

(b) State and prove Jordan-Holder theorem. 7

(c) Let G be a group of order 108. Show that there exist a normal subgroup of order 27 or 9. 6

6. (a) State and prove Sylow's second theorem. 7

(b) Let G be a finite group of order p^n, where p is prime and $n > 0$. Then prove that $Z \cap N$ is nontrivial for any nontrivial normal subgroup N of G. 7

(c) Show that a simple group is solvable if and only if it is cyclic. 6

SECTION—D

7. (a) Find all ideals in $\mathbb{Z}$ and also in $\mathbb{Z}_{10}$. 5

(b) If R is a ring with unity, then each maximal ideal is prime. Is converse true ? Justify. 6

(c) Let F be a field. Then characteristic of F is either 0 or a prime number p. 5

(d) Define idempotent and find the idempotents of ring $\mathbb{Z}_{12}$. 4

8. (a) Show that there exist a ring homomorphism $f : \mathbb{Z}_m \to \mathbb{Z}_n$ if and only if $n \mid m$. 6

(b) Prove that the ideal $<x^3 + x + 1>$ in the polynomial ring $Z_2[x]$ over $\mathbb{Z}_2$ is a prime ideal. 6

(c) Define integral domain and show that a finite integral domain is a division ring. 8